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THE TRISPECTRUM OF THE 4 YEAR COBE DMR DATA

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ABSTRACT

We propose an estimator for the trispectrum of a scalar random field on a sphere, discuss its geometrical and statistical properties, and outline its implementation. By estimating the trispectrum of the 4 yr COBE Differential Microwave Radiometer experiment data (in HEALPix pixelization), we find new evidence of a non-Gaussian signal associated with a known systematic effect. We find that by removing data from the sky maps for those periods of time perturbed by this effect, the amplitudes of the trispectrum coefficients become completely consistent with predictions for a Gaussian sky. These results reinforce the importance of statistical methods based in harmonic space for quantifying non-Gaussianity.

Subject headings: cosmic microwave background — cosmology: observations

1. INTRODUCTION

The cosmic microwave background (CMB) is the cleanest window on the origin of structure in the very early universe. A complete description of the statistical properties of cosmological fluctuations at a redshift of \( z \approx 1000 \) affords us an essential window on the origin of structure in the very early universe. A powerful discriminator between different models of structure formation.

Most analyses of CMB data to date have focused on the angular power spectrum and its sensitivity to various parameters of cosmological theories. Some work has been done on the estimation of the three-point correlation function and its analog in spherical harmonic space, with intriguing results (Heavens 1998; Ferreira, Magueijo, & Górski 1998; Magueijo 2000; Banday, Zaroubi, & Górski 2000). It is the purpose of this Letter to propose a method for estimating the four-point correlators, and (3) they satisfy the appropriate symmetries under rotations.

In this section we wish to construct a set of quantities for estimating the trispectrum of a random field on the sphere. The temperature anisotropy in a given direction on the celestial sphere, \( T(\mathbf{n}) \), can be expanded in terms of spherical harmonic functions, \( Y_{lm}(\mathbf{n}) \):

\[
T(\mathbf{n}) = \sum_{lm} a_{lm} Y_{lm}(\mathbf{n}).
\]

2. THE ESTIMATOR

For any theory of structure formation, the \( a_{lm} \) coefficients are a set of random variables; we shall restrict ourselves to theories that are statistically homogeneous and isotropic. In this case, we can define the power spectrum \( C_l \) of the temperature anisotropies by \( \langle a_{lm} a_{lm}^* \rangle = C_l \delta_{ll} \).

We now seek to construct a set of tensors that are geometrically independent, describe their statistical properties for a Gaussian random field, and then discuss the practical issue of their implementation. Given a set of \( a_{lm} \), we wish to find the index structure of the set of four-point correlators such that (1) they are rotationally invariant, (2) they form a complete basis (preferably orthonormal) of the whole space of admissible four-point correlators, and (3) they satisfy the appropriate symmetries under interchanges of \( m \)- and \( l \)-values. We shall restrict ourselves to the case in which \( l_1 = l_2 = l_3 = l_4 = l \). Furthermore, throughout this section we keep \( l \) fixed. We determine the tensor \( T \) such that

\[
\langle a_{lm} a_{lm} a_{lm} a_{lm} \rangle = \sum_{\mu=0}^n T_{\mu l} T_{\mu l} T_{\mu l} T_{\mu l},
\]

where \( n = \text{int}(l/3) \) (due to reflection, permutation, and rotational symmetry). The \( T_{\mu l} \)-values are then the components of the tri-

\[
\text{(2)}
\]
spectrum that we wish to estimate. The explicit form of $T$ is

$$T_{m_1m_2m_3m_4} = \sum_{\alpha=0}^{\infty} \int_{l_1}^{l_2} T_{m_1m_2m_3m_4}^\alpha \, dl,$$

(3)

$$\bar{T}_{m_1m_2m_3m_4} = \sum_{M=-2a}^{2a} (-1)^M \left( \begin{array}{ccc} l & l & 2\alpha \\ m_1 & m_2 & M \end{array} \right) \times \left( \begin{array}{ccc} 2\alpha & l & l \\ -M & m_3 & m_4 \end{array} \right) + \text{inequivalent permutations},$$

(4)

where the matrices in parentheses are the Wigner 3J symbols. The $T^\alpha$ are not orthogonal and satisfy

$$\bar{T}_{m_1m_2m_3m_4} \bar{T}_{m_1m_2m_3m_4} = \frac{3}{4\alpha + 1} \delta_{ab} + 6 \left\{ \begin{array}{ll} 1 & 2\alpha \\ l & l/2 \beta \end{array} \right\}_a$$

(5)

(where summation over the $m_i$ is assumed), which has a rank of $n + 1$. The matrix $L_a$ in equation (3) is a rectangular matrix (with a triangular subblock) with $n + 1$ columns and $l + 1$ rows. It is constructed through a Gram-Schmidt procedure by subtracting for each $\alpha$ (starting from $\alpha = 0$) the projection onto all $a' < a$ and then normalizing the result. The $\alpha = 0$ (and hence $\alpha = 0$) tensor is proportional to the Gaussian contribution. This can be easily seen given that for $\alpha = 0$ the Wigner 3J symbols are simply Kronecker $\delta$ symbols in the corresponding indices. The remaining $a > 0$ terms contain therefore no Gaussian signal and quantify the non-Gaussian part of the trispectrum.

The $T$-values are orthonormal and can be used to construct an estimator for $T$ from a realization of $a_{lm}$:

$$\hat{T}_{l,a} = T_{m_1m_2m_3m_4} a_{lm} a_{lm} a_{lm} a_{lm}^\ast,$$

(6)

For a Gaussian random field we expect $\sigma^2[\hat{T}_{l,0}] \gg \sigma^2[\hat{T}_{l,a}]$ for $a > 0$, where $\sigma^2[A]$ denotes the variance of the random variable $A$ and $\hat{T}_{l,a}$ is simply the square of the minimum variance estimator of the $C_l$. One finds that $\langle \hat{T}_{l,a} \rangle = 0$ and $\sigma^2[\hat{T}_{l,a}] = 24C_l^{-4}$ for all $a > 0$.

To show that the $\hat{T}_{l,a}$ constitute a family of minimum variance estimators, we construct a linear combination of the estimators,

$$\hat{T}_{m_1m_2m_3m_4} = \sum_{a=0}^{\infty} c_a \bar{T}_{m_1m_2m_3m_4}^\alpha,$$

and minimize the function

$$\sigma^2[c_a, \lambda] = \langle \left( \bar{T}_{m_1m_2m_3m_4} a_{lm} a_{lm} a_{lm} a_{lm}^\ast \right)^2 \rangle - \langle \bar{T}_{m_1m_2m_3m_4} a_{lm} a_{lm} a_{lm} a_{lm}^\ast \rangle^2 - \lambda C_l^{-4} \left( \bar{T}_{m_1m_2m_3m_4} \bar{T}_{m_1m_2m_3m_4} - 1 \right),$$

(8)

where summation over all $m_i$ is implied. The last term, a Lagrange multiplier, ensures that $T^\alpha$ is normalized. We solve $\partial_a \sigma^2[c_a, \lambda] = \partial_a \sigma^2[c_a, \lambda] = 0$ to find a set of two equations:

$$(24I + 72A_l)^{ab} c_b + \lambda c_b = 0,$$

$$c^2 = 1,$$

(9)

This is an eigenvector equation where for a given eigenvector $c$, the eigenvalue $\lambda$ gives the expected variance of the estimator. Of the $n + 1$ eigenvalues, one is large and has an eigenvector proportional to $\hat{T}_{l,a}$. The remaining eigenvalues have an amplitude of $\lambda = 24$, and each eigenvector is $A_l^{ab} \hat{T}_{l,a}$ for $a > 0$.

Note that we can relate our parameterization to the one proposed in Hu (2001); if we reexpress equation (2) as

$$\langle a_{lm} a_{lm} a_{lm} a_{lm} \rangle = \sum_{a=0}^{\infty} \hat{T}_{l,a} \bar{T}_{m_1m_2m_3m_4}^\alpha,$$

(10)

where $T_{l,a} = L^{-1/n}_{a} \hat{T}_{l,a}$, then $Q_l^{ab}$ as defined in equation (15) of Hu (2001) can be written as

$$Q_l^{ab}(2\alpha) = \hat{T}_{l,a} + (4\alpha + 1) \sum_{\beta} \left\{ \begin{array}{ll} 1 & 2\alpha \\ l & l/2 \beta \end{array} \right\} \hat{T}_{l,\beta}.$$
As an application of the formalism described in § 2, we estimate the trispectrum of the co-added 53 and 90 GHz COBE DMR 4 yr sky maps in HEALPix format (Górski, Hivon, & Wandelt 1999). The resolution of the maps is $N_{\text{side}} = 64$, or 49,152 pixels.

We do not extend our analysis beyond $l_{\text{max}} = 20$ since the signal-to-noise ratio is poor for higher $l$. Hence, the maximal number of independent non-Gaussian estimators for the trispectrum is $\text{int}(l_{\text{max}}/3) = 6$. We set the pixels in the extended Galactic cut (Banday et al. 1997) to zero and subtract the residual monopole and dipole of the resulting map. After convolving the maps with spherical harmonics to extract a set of $\alpha_{lm}$-values for $l \leq 20$, we then apply equation (6). To validate our software, we have estimated the bispectrum of the COBE DMR 4 yr sky data repixelized in the HEALPix format (denoted by EC for convenience) and reproduced the results of Ferreira et al. (1998), in particular the strong non-Gaussian signal present at $l = 16$. When an equivalent map, from which that part of the DMR time stream contaminated by the “eclipse effect” is removed (denoted NEC), is subsequently analyzed, we also reproduce the results of Banday et al. (2000), namely, that the non-Gaussian signal is no longer detected. For our subsequent analysis we will present the trispectra of both the EC and NEC data.

One of our primary concerns is to compare our results with the assumption that the CMB sky measured by COBE DMR is Gaussian. To do so, we generate 10,000 full-sky maps at the same resolution using a scale-invariant power spectrum normalized to $Q_{\text{obs}} - ps = 18 \, \mu$K (Górski et al. 1998). We convolve each map with the DMR beam and add uncorrelated pixel noise with rms amplitude $\sigma_p = 15.95$ mK/($N_{\text{obs}})^{1/2}$, (where $N_{\text{obs}}$ is the number of times a given pixel was observed); we then subject the synthetic map to the same procedure as the original data.

Figure 1 shows the trispectra of the DMR data together with Gaussian 95% confidence limits. Instead of the “raw” estimator (6), we prefer to use the normalized trispectrum, $\tau_{ab}^{\alpha} = \hat{T}_{ab} C_i^2$ for $a > 1$ (where $C_i = [1/(2l + 1)][\sum \alpha_{lm}^2]$, thus effectively removing the dependence on the power spectrum. This prevents fluctuations in the power spectrum from introducing spurious signals and from masking real non-Gaussianities. Figure 1 shows that in this case, most values fall within the 95% confidence lines and demonstrate the scatter expected for a Gaussian random field.

Of particular interest is the value of the normalized $\tau^{(3)}$ at $l = 16$ in Figure 1. One finds that 99.9% of the Gaussian models in the EC case have a smaller $\tau^{(3)}$ than the measured one. This is clearly a manifestation of the non-Gaussianity found in Ferreira et al. (1998), which is highly localized in $l$ space. However, if we estimate $\tau^{(3)}$ for the NEC we find that it falls comfortably within the 95% confidence limits. This leads us to believe that this detection of non-Gaussianity results from the eclipse effect, consistent with the hypothesis of Banday et al. (2000).

We construct a goodness of fit for our statistic. In Ferreira et al. (1998), a modified $\chi^2$ was constructed that took into account the non-Gaussian distribution of each method: as above, the distribution of each estimator for a Gaussian sky was constructed and used as an approximate likelihood function to evaluate the goodness of fit. One shortcoming of such a method was that correlations between the estimates for different $l$-values were discarded. To include them, we use the Gaussian ensemble of data sets to derive the expectation values $(\langle \cdot \rangle_G)$ and the covariance matrix $C$ for both the power spectrum, $C$, and all seven trispectrum estimators, $\tau_{ab}^{(3)}$. We proceed to calculate the $\chi^2$-value for the estimator $E$ and the data set $D$,

$$\chi^2(E, D) = \sum_{l, p} \left[ (E_{l} - \langle E \rangle_G) C^{-1} (E_{l} - \langle E \rangle_G) \right],$$

using as data sets the EC and NEC data. Finally, we use another 10,000 Gaussian realizations to estimate the expected distribution of the $\chi^2$ for both the EC and the NEC data.

For all normalized non-Gaussian trispectrum estimators ($\tau_{b}^{(3)}$ to $\tau_{b}^{(6)}$), we find that 94% of the Gaussian models have a smaller $\chi^2$ than the EC data, as can be seen in Figure 2. As expected, the main contribution to the $\chi^2$ for the EC data stems from $\tau_{b}^{(3)}$ at $l = 16$; indeed, this is the only normalized trispectrum estimator that exhibits any significant non-Gaussianity, in this case at about 99.9%. If we use the NEC data, the detection vanishes. In this case, 60% of all Gaussian models have a lower $\chi^2$ when computed over all six trispectrum estimators (83% for $\tau_{b}^{(3)}$ alone). Hence, the NEC data is compatible with Gaussianity.

4. DISCUSSION

In this Letter, we have derived an estimator for the trispectrum of a scalar random field on the sphere. Application of this estimator, normalized by the power spectrum (a procedure adopted in Ferreira et al. 1998 for the bispectrum; see also Komatsu et al. 2002 for a detailed discussion), to the COBE DMR data provides evidence for non-Gaussianity at the 94% confidence level. As in the case of the bispectrum, the signal is mainly present in the $l = 16$ multipole (and the $\tau_{b}^{(3)}$ estimator here). However, when data is excluded to correct for the eclipse effect, the non-Gaussian behavior is removed, allowing us to...
conclude that the non-Gaussianity present in the uncorrected sky maps is not cosmological in origin.

The detection of a signal that is so strongly localized in $l$ space provides convincing support to our contention that the trispectrum is an important and sensitive probe of non-Gaussianity in the frequency (scale) domain. It affords complementary information to the bispectrum since it is an even moment and, despite the higher computational effort required, has the obvious advantage in that it can probe all values of $l$, not just the even ones.

Interestingly enough, from a theoretical perspective there may be some possible sources of non-Gaussianity for which the trispectrum provides a far more sensitive test than the bispectrum. In many cases, a given moment of the $a_{lm}$-values can be expressed as the projection of a cosmological field. If that field is vector-like in nature (as in the case of the Doppler effect or the Ostriker-Vishniac effect and its non-linear extensions), any odd moment may suffer from the Sunyaev-Kaiser cancellation, where the integral of a given wavenumber $k$ over a smoothly varying projection function with width $\sigma$ tends to suppress the moment by a factor on the order of $1/(\sigma k)^{3/2}$ (Sunyaev 1978; Kaiser 1984; Scannapieco 2000). For even moments one can always construct a scalar component that will not be subject to this cancellation. Such a tool will be of great use in the analysis of the data sets from the Microwave Anisotropy Probe and Planck Surveyor satellites.

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Note added in proof.—E. Komatsu investigates the trispectrum of COBE DMR data in his Ph.D. thesis. His conclusions agree with ours, namely, that the COBE data is consistent with Gaussian initial fluctuations.

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Fig. 2.—$\chi^2$ distribution of the Gaussian models (histogram) and actual data value (dotted line) for the EC (top row) and NEC (bottom row) data sets. The left two graphs show $\tau^{(3)}$, which contains the main contribution to the non-Gaussian signal, while the right two graphs show the total $\chi^2$ over all six non-Gaussian estimators, $\tau^{(1)}$ to $\tau^{(6)}$.