WIMP dark matter and the QCD equation of state

Hindmarsh, Mark and Philipsen, Owe (2005) WIMP dark matter and the QCD equation of state. Physical Review D, 71 (8). 087302. ISSN 1550-7998

This version is available from Sussex Research Online: http://sro.sussex.ac.uk/25906/

This document is made available in accordance with publisher policies and may differ from the published version or from the version of record. If you wish to cite this item you are advised to consult the publisher's version. Please see the URL above for details on accessing the published version.

Copyright and reuse:
Sussex Research Online is a digital repository of the research output of the University.

Copyright and all moral rights to the version of the paper presented here belong to the individual author(s) and/or other copyright owners. To the extent reasonable and practicable, the material made available in SRO has been checked for eligibility before being made available.

Copies of full text items generally can be reproduced, displayed or performed and given to third parties in any format or medium for personal research or study, educational, or not-for-profit purposes without prior permission or charge, provided that the authors, title and full bibliographic details are credited, a hyperlink and/or URL is given for the original metadata page and the content is not changed in any way.

http://sro.sussex.ac.uk
Dark matter of weakly interacting massive particles and the QCD equation of state

Mark Hindmarsh1 and Owe Philipsen1,2

1Department of Physics & Astronomy, University of Sussex, Brighton BN1 9QH, United Kingdom
2Institut für Theoretische Physik, Westfälische Wilhelms-Universität Münster, 48149 Münster, Germany

Abstract

Weakly Interacting Massive Particles (WIMPs) of mass \( m \) generically freeze out at a temperature \( T_f \approx m/25 \) GeV, and so for typical masses in the range 10 – 1000 GeV, \( T_f \) lies between 400 MeV and 40 GeV. The equation governing the relic density depends on both energy and entropy densities, and so the WIMP relic density is sensitive to the equation of state of the Universe in this temperature range. It is normally supposed that, above the QCD confinement critical temperature of \( T_c \approx 200 \) MeV, the plasma is weakly interacting because of asymptotic freedom, and can be treated as an ideal gas, and this assumption is built into at least two of the best-known SUSY WIMP relic density packages, DarkSUSY [2] and MicrOMEGAs [3].

However, intensive nonperturbative studies and experiments revealed that the high-temperature QCD plasma still departs significantly from an ideal gas at temperatures several orders of magnitude higher than \( T_c \). It is therefore worth re-examining the equation of state for the Universe in this critical regime, and qualitative investigations have been made before [4]. Using recent progress in both the deconfined [5] and confined [6] phases, we construct an improved equation of state and investigate the consequences for the WIMP relic density in one of the dark matter packages, DarkSUSY. DarkSUSY currently uses an equation of state [7] based on work in Refs. [8,9]. Replacing it with the improved version gives upward corrections of over 2% to the relic densities in a set of benchmark mSUGRA models [10], and larger effects for models with lower \( T_f \).

Although small, this correction is becoming significant in the new era of precision cosmology: the cold dark matter (CDM) density is determined to better than 10% in single-field inflation models from a combination of the Cosmic Microwave Background (CMB) angular power spectrum and observations of the galaxy power spectrum by 2dF [11] and by SDSS [12]. Planck promises to do much better, with one estimate giving a determination to better than 1% [13].

The calculations we use still involve simplifications and systematic errors, forcing us to model the equilibrium pressure \( p(T) \) to some extent, as described below. Estimating our systematic errors on the QCD equation of state to be on the order of 10% near \( T_c \), we show how they propagate into uncertainties in the range of 0.5 – 1% in the relic density of WIMPs. Thus, the quest for new physics in observational cosmology depends on further improvements in the accuracy of the quantitative understanding of the QCD equation of state.

There has been much effort in calculating the pressure \( p(T) \) of an SU(\( N_c \)) gauge theory with \( N_f \) fundamental fermions at temperature \( T \), as reviewed in Refs. [14 –18]. Perturbative expansions in the coupling constant \( g \) of quantities such as the pressure converge badly, and particularly for a strongly-coupled theory like QCD. Strictly perturbative expansions, even when expanded to \( O(g^6 \ln g) \) [5], seem to converge well only at remarkably high temperatures, above \( 10^3 T_c \), in sharp contradiction to the ideal gas assumption usually made in cosmology.

In the high-temperature regime, progress has been made using perturbative finite-temperature dimensional reduction (DR) (see [5] and references therein). By constructing a sequence of effective theories for the scales \( 2\pi T, gT \) and \( g^2 T \), the last of which has to be treated nonperturbatively, one can get results for arbitrary \( N_f \) of massless quarks applicable down to a few times \( T_c \). By fitting for the nonperturbative and as yet unknown \( O(g^6) \) coefficient, the authors of Ref. [5] were able to match their calculated

DOI: 10.1103/PhysRevD.71.087302 PACS numbers: 95.35.+d, 12.38.Mh
pressure reasonably well to pure-glue lattice data near the critical temperature.

Around the transition, there now exist lattice calculations for the pressure and energy density for $N_f = 0, 2, 3$ degenerate quark flavours, as well as first data for $N_f = 2 + 1$, i.e. two light and one heavier flavour. The pseudo-critical temperatures $T_c(N_f)$ (defined as the peak of a susceptibility) are currently given as $T_c(0) = 271 \pm 2$ MeV, $T_c(2) = 173 \pm 8$ MeV and $T_c(3) = 154 \pm 8$ MeV [19]. It has to be stressed, however, that only the pure-glue case has been extrapolated to the continuum. Based on experience with that theory, the dynamical fermion results are estimated to display a systematic error of about 15% at the currently available lattice spacing. Corrections due to quark masses deviating from the physical ones appear to be negligible in comparison [19].

Finally, below the phase transition the hadronic resonance gas model, which treats the plasma as an ideal gas of mesons, baryons and their excited states, matches reasonably well to lattice data [6].

We now review relic density calculations. Consider a particle of mass $m$ and number density $n$, undergoing annihilations $XX \rightarrow \ldots$ with total cross-section $\sigma$, assumed to be a typical weak interaction cross-section, proportional to $G_F^2$. Then [20]

$$\dot{n} + 3Hn = -\langle \sigma \nu_{\text{mol}} \rangle (n^2 - n_{\text{eq}}^2),$$

where $n_{\text{eq}}$ is the equilibrium number density, $H = \dot{a}/a$ is the Hubble parameter, and $\nu_{\text{mol}}$ is the Møller velocity which is a relativistic generalization of the relative velocity of the annihilating particles [20].

In order to solve the equation it is convenient to convert the time variable to $x = m/T$, and to measure the relic abundance in terms of $Y = n/s$, where $s$ is the entropy density, related to the energy density $\rho$ and the pressure $p$ through $s = (\rho + p)/T$. If the total entropy $S = sa^3$ is conserved, then we can write

$$\frac{dY}{dx} = \frac{1}{3H} \frac{ds}{dx} \langle \sigma \nu_{\text{mol}} \rangle (Y^2 - Y_{\text{eq}}^2).$$

This adiabaticity assumption is violated if the QCD transition is first order, but it is most likely to be a crossover transition at the low chemical potentials which are relevant for the early Universe [17,18].

We define effective numbers of degrees of freedom for the energy and entropy densities ($g_{\text{eff}}(T)$ and $h_{\text{eff}}(T)$ respectively) through

$$\rho = \frac{\pi^2}{30} g_{\text{eff}}(T) T^4, \quad s = \frac{2\pi^2}{45} h_{\text{eff}}(T) T^3.$$  \hspace{1cm} (3)

Using the Friedmann equation $H^2 = 8\pi G \rho / 3$, one finds an approximate solution

$$Y_0 \approx \left(\frac{45}{\pi}\right)^{1/2} \frac{1}{mM_p(\sigma \nu_{\text{mol}}) T_c} \frac{x_i}{g_{\text{eff}}^2(T_c)},$$

where $T_f$ is the freeze-out temperature, defined to be the temperature at which the relic abundance is a certain factor (taken to be 2.5) above the equilibrium abundance. We see that the relic density depends on the parameter

$$g_{\text{eff}}^{1/2}(T) = \frac{h_{\text{eff}}^{1/2}}{g_{\text{eff}}^{1/2}} \left(1 + \frac{T}{3} \frac{d \ln h_{\text{eff}}}{dT} \right).$$

It is through this parameter that the QCD equation of state influences the relic density.

In an ideal gas at temperature $T$, a particle of mass $m_i = x_i T$ contributes to $g_{\text{eff}}$, $h_{\text{eff}}$ the amounts

$$g_{i,\text{eff}} = \frac{\rho_i}{\rho_0} = \frac{15}{\pi^3} \int_{x_i}^\infty \frac{(u^2 - x_i^2)^{1/2}}{e^u - 1} u^2 du,$$  \hspace{1cm} (6)

$$h_{i,\text{eff}} = \frac{s_i}{s_0} = \frac{45}{12 \pi^4} \int_{x_i}^\infty \frac{(u^2 - x_i^2)^{1/2}}{e^u - 1} (4u^2 - x_i^2) du,$$  \hspace{1cm} (7)

where $\rho_0$ and $s_0$ are the energy and entropy densities for a free massless boson.

Interactions correct the ideal gas result, and $g_{\text{eff}}$, $h_{\text{eff}}$ have to be extracted from calculations of the energy and entropy, Eqs. (3). Note that in the relevant temperature range 40 to 0.4 GeV, 16 out the 18 bosonic degrees of freedom are coloured, with the coloured fermionic degrees of freedom dropping from 60/78 to 36/50. The dominant corrections are therefore expected to come from the coloured sector of the standard model, weak corrections are moreover suppressed by the boson masses and negligible.

The relic density codes DarkSUSY and MicrOMEGAs use identical equations of state, developed in Refs. [7–9]. Below $T_c$, the hadronic degrees of freedom are modeled by an interacting gas of hadrons and their resonances, while above $T_c$, the quarks and gluons are taken to interact with a linear potential $V_\rho(r) = Kr$, with a phenomenologically motivated value $K = 0.18$ GeV$^2$, derived from the slope of Regge trajectories. In this model, the pressure is already very close to ideal at temperatures above 1.6 GeV. All other standard model particles are free.

In this work we also take the ideal gas contributions for the particles of the standard model, with masses given by the Particle Data Group central values [21]. In the confined phase quarks and gluons are replaced by hadronic models described below. In the deconfined phase, the contribution to the pressure of the colored degrees of freedom is scaled by a function $f(T)$, defined to be the ratio between the true QCD pressure $p(T)$ and the Stefan-Boltzmann result $p_{SB}$ for the same theory, $f(T) = p(T)/p_{SB}(T)$. This correction factor is derived from lattice [19] and perturbative [5] calculations for $N_f = 0$, and uses an approximate universality in the pressure curves for different $N_f$ observed by Karsch et al. [19]. Near the transition, the lattice-derived curves for $f(T)$ have the approximate form $f(T, N_f) = f_{\text{QCD}}(T/T_c(N_f))$, where $T_c(N_f)$ is the critical temperature.
for the theory with \( N_f \) light fermion flavours. Besides this, there appears to be only negligible \( N_f \)-dependence within the current numerical accuracy. We are therefore motivated to neglect quark mass effects, take the \( N_f = 0 \) lattice data (in the continuum limit) and the \( N_f = 0 \) DR formula of Kajantie et al. [5], and scale the temperature dependence by \( T_{0}(3)/T_{0}(0) \). The correction factors are matched at 1.2 GeV using the undetermined \( O(g^6) \) parameter, which we take to be 0.6755, close to the value 0.7 used in Ref. [5].

At higher temperatures one crosses the \( c \) and \( b \) mass thresholds. However, for \( N_f > 3 \) the \( O(g^6) \) fitting parameter is unknown, and the appropriate critical temperatures are also unknown. Hence we believe that scaling the \( N_f = 0 \) result is the best we can do at present. We will discuss later how improvements can be made.

In the confined phase we label our model equations of state (EOS) A, B and C. EOS A ignores hadrons completely, as the lattice shows that \( f(T) \) very rapidly approaches zero below \( T_c \). EOS B and C model hadrons as a gas of free mesons and baryons. It was noted in Ref. [6] that such a gas, including all resonances, gives a pressure which fits remarkably well to the \( N_f = 2 + 1 \) lattice results although, as the authors themselves point out, this result should be treated with caution as the simulations are not at the continuum limit. We include all resonances listed in the Particle Data Group’s table mass_width_02.mc [21].

We make a sharp switch to the hadronic gas at a temperature \( T_{HG} \). For our EOS B we take \( T_{HG} = T_c = 154 \) MeV, and for EOS C we take \( T_{HG} = 200 \) MeV and \( T_c = 185.5 \) MeV, values chosen to give as smooth a curve for \( h_{\text{eff}} \) as possible. The effects of these equations of state on the relic densities turn out to differ by less than 0.3% in the relevant temperature interval, so in the following we concentrate on EOS B.

Before presenting our results we note that \( \Omega h^2 \) is directly proportional to the entropy density today \( s_0 = (2\pi^2/45)h_{\text{eff}}(T_\gamma)T_\gamma^3 \), and that this must be determined as accurately as possible. The photon temperature \( T_\gamma \) is \( 2.725 \pm 0.001 \) is very accurately measured [22], but the contribution from neutrinos requires a separate freeze-out calculation. Recent work [23] gives \( h_{\text{eff}}(T_\gamma) = 3.9172 \). We find that taking freeze-out temperatures to be 3.5 MeV for \( \nu_\mu \) and \( \nu_\tau \) and 2 MeV for \( \nu_\tau \), as recommended in Ref. [7], gives \( h_{\text{eff}}(T_\gamma) = 3.9138 \), which is accurate enough for our purposes. DarkSUSY uses \( h_{\text{eff}}(T_\gamma) = 3.9139 \).

In Figs. 1 we plot for our EOS B the effective numbers of degrees of freedom \( h_{\text{eff}}(T) \) and \( g^*_1(T) \) defined in Eqs. (3) and (5), compared with those used in DarkSUSY [2] and MicrOMEGAs [3]. The spike in our \( g^*_1(T) \) is an artefact of the matching of the scaled lattice data and the hadronic equation of state (i.e. the first derivative jumps). Since for physical QCD the transition is a smooth crossover, this spike is unphysical. However, it has no noticeable influence on the freeze-out of WIMPs at higher temperatures.

In Table I we exhibit the effect of the new equation of state on the density of relic neutralinos \( \chi \), for the mSUGRA models used to test DarkSUSY in the standard distribution [24]. We find changes of about 1.5–3.5%. In order to quantify the effect of uncertainty in the lattice data, we introduce two new models B2 and B3, which are constructed by scaling the lattice curve by 0.9 and 1.1, respectively, and then adjusting the \( O(g^6) \) parameter in the DR pressure curve so that it meets the scaled lattice curve at \( T = 4.43T_c \). Thus a 10% uncertainty in the lattice pressure curve translates to an uncertainty in the relic density in the range 0.5–1%. Note that the lowest freeze-out temperature in the table is about 4 GeV. This corresponds to more than 20\( T_c \), where the QCD corrections to \( g^*_1 \) are around 5%. Evidently, WIMPs with freeze-out temperatures closer to \( T_c \) would be affected more strongly.

![Image](https://example.com/image.png)
The production of sterile neutrinos is also affected by the details of the QCD transition [4].

To conclude, by updating the equation of state of the standard model in the light of recent developments in high-temperature QCD, we have found differences in WIMP relic density calculations of a few per cent for the benchmark models of Ref. [10], a small but not insignificant effect. These models freeze out well above $T_c$, and so it is likely that there are other models exhibiting greater changes. In single-field inflation models, the 68% confidence limit on the dark matter density is already $\Omega_c h^2 = 0.127 \pm 0.017$ from the WMAP First Year data [11], while Planck has been estimated to be able to reach a level of $\Delta \Omega_c h^2 = 0.0011$ [13]. In order to benefit from this accuracy, it has to be matched by that of theoretical calculations of the QCD equation of state. To obtain an accuracy of 1% we estimate that we require lattice data accurate to 10% at about $T \approx 4T_c$, for realistic quark masses, and in the continuum limit. Further progress is possible: the $O(g^4)$ term is in principle calculable but requires a 4-loop computation in lattice perturbation theory [5]. Moreover, the $b$ and $c$ quark mass thresholds may affect the QCD corrections. We also note that the equation of state at a few $T_c$ is also experimentally accessible in current and future heavy ion experiments, which gives them a direct cosmological motivation.

Finally, given a sufficiently accurate $g_*$, we speculate that with a combination of measurements of the neutralino mass and cross-sections in direct detection experiments and at LHC and the ILC, the effective number of degrees of freedom in the early Universe at a few GeV will become experimentally accessible, motivating attempts to improve further on this work.

We are indebted to F. Karsch and J. Engels for lattice data, to M. Laine for providing details from Ref. [5], and to P. Gondolo and F. Boujdjema for discussions. We acknowledge support from PPARC (OP) and from the European Network for Theoretical Astroparticle Physics (ENTApP), member of ILIAS, EC contract number RII-CT-2004-506222 (MH).

<table>
<thead>
<tr>
<th>Model</th>
<th>$m_\chi$/GeV</th>
<th>$T_c$/GeV</th>
<th>$\Omega_c h^2$ (DS)</th>
<th>$\Omega_c h^2$ (here)</th>
<th>$\Delta$(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A'</td>
<td>242.83</td>
<td>9.8</td>
<td>0.0929</td>
<td>0.0948(54)(42)</td>
<td>2.0(2.6)(1.4)</td>
</tr>
<tr>
<td>B'</td>
<td>94.88</td>
<td>4.1</td>
<td>0.1213</td>
<td>0.1242(56)(31)</td>
<td>2.4(3.6)(1.5)</td>
</tr>
<tr>
<td>C'</td>
<td>158.09</td>
<td>6.5</td>
<td>0.1149</td>
<td>0.1174(83)(65)</td>
<td>2.2(2.9)(1.5)</td>
</tr>
<tr>
<td>D'</td>
<td>212.42</td>
<td>8.6</td>
<td>0.0864</td>
<td>0.0882(88)(76)</td>
<td>2.0(2.7)(1.4)</td>
</tr>
<tr>
<td>G'</td>
<td>147.98</td>
<td>6.2</td>
<td>0.1294</td>
<td>0.1323(33)(13)</td>
<td>2.2(3.0)(1.4)</td>
</tr>
<tr>
<td>H'</td>
<td>388.38</td>
<td>16.0</td>
<td>0.1629</td>
<td>0.1662(71)(53)</td>
<td>2.0(2.6)(1.5)</td>
</tr>
<tr>
<td>I'</td>
<td>138.08</td>
<td>5.8</td>
<td>0.1319</td>
<td>0.1351(62)(40)</td>
<td>2.4(3.2)(1.6)</td>
</tr>
<tr>
<td>J'</td>
<td>309.17</td>
<td>12.6</td>
<td>0.0966</td>
<td>0.0984(90)(79)</td>
<td>2.0(2.5)(1.4)</td>
</tr>
<tr>
<td>K'</td>
<td>554.19</td>
<td>22.9</td>
<td>0.0863</td>
<td>0.0883(88)(78)</td>
<td>2.3(3.0)(1.8)</td>
</tr>
<tr>
<td>L'</td>
<td>180.99</td>
<td>7.5</td>
<td>0.0988</td>
<td>0.1011(18)(03)</td>
<td>2.3(3.0)(1.5)</td>
</tr>
</tbody>
</table>