ANNOTATIONS OF AN EXAMPLE THAT MARKOV PROCESS HAS NO STRONG MARKOV PROPERTY

TANG RONG

Abstract. In this paper, we discuss an incorrect example that a Markov process does not satisfy strong Markov property, and analyzes the reason of mistake. In the end, we point out it is not reasonable to define strong Markov property by using transition probability functions since transition probability functions might not be one and only.

1. Introduction

People always believe that there exists this kind of Markov processes such that they have no strong Markov property, because someone has given an example that Markov process does not satisfy strong Markov property. We cannot help but think this example is of great importance, since this example produce extensive (negative) influence for research Markov process even stochastic process. The following we will give the example, a Markov process does not satisfy strong Markov property, in some reference, but we will point out the example is false.

2. An example and its proof in some reference

Example 2.1. Let \( \{B_t(\omega) ; t \geq 0 \} \) is a Brown motion. Put

\[
A_t(\omega) = \begin{cases} 
  B_t(\omega) & \text{if } B_0(\omega) \neq 0, \\
  0 & \text{otherwise}
\end{cases} = B_t(\omega)X_{\{B_0 \neq 0\}}(\omega);
\]

\[
P(t; x, B) = \left\{ \begin{array}{ll}
  \int_{B \in \mathcal{B}(\triangle = \mathcal{B}((-\infty, \infty)))} e^{-\frac{(x-y)^2}{2t}} d\gamma & \text{if } x \neq 0, \\
  X_B(0) & \text{if } x = 0,
\end{array} \right.
\]

Then \( \{A_t(\omega); t \geq 0\} \) is a Markov process with transition probability functions \( P(t; x, B) \), but it has no strong Markov property. Here \( X_{\{B_0 \neq 0\}}(\omega) \) is a indicator function relative to set \( \{B_0 \neq 0\} \).

Note. Intuitively, if we fix the trajectory of \( A_t(\omega) \), then the trajectory of \( A_t(\omega) \) is the same as \( B_t(\omega) \) when the initial state of \( A_t(\omega) \) is not 0, and the trajectory of \( A_t(\omega) \) is forever 0 when the initial state of \( A_t(\omega) \) is 0. So \( A_t(\omega) \) is not homogeneous Markov process. For if it is homogeneous Markov process, then for any \( \omega \in \Omega, A_t(\omega) \) never leave 0 after hitting 0 by homogeneity, but on the other hand, for any \( \omega \) with \( A_0(\omega) \neq 0 \), \( A_t(\omega) \) must leave 0 after hitting 0 by the definition of \( A_t(\omega) \). This is a contradiction.

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Proof. Let $F_t = \mathcal{F}(B_u; u \leq t)$. Obviously, $A_t$ is $\mathcal{F}_t$-measurable. For any $B \in \mathcal{B}$ and $t, s > 0$, we have

$$E(X_B(A_{t+s})|\mathcal{F}_s) = E(X_B(B_{t+s})X_{\{B_0 \neq 0\}}(\omega)|\mathcal{F}_s) + E(X_B(0)X_{\{B_0 = 0\}}(\omega)|\mathcal{F}_s)$$

$$= X_{\{B_0 \neq 0\}}(\omega) \int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} dy + X_{\{B_0 = 0\}}(\omega) X_B(0)$$

$$= X_{\{A_0 \neq 0\}}(\omega) \int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} dy + X_{\{A_0 = 0\}}(\omega) X_B(A_s)$$

$$(2.1)$$

$$+ X_{\{A_0 = 0, B_0 \neq 0, B_s = 0\}}(\omega) \left[ \int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} dy - X_B(0) \right]$$

$$= P(t; A_s(\omega), B), \text{ } P.a.e.,$$

where the last equality follows from that the last term of (2.1) is equal to 0 almost everywhere. So $A_t$ is a Markov process with transition probability functions $P(t; x, B)$, but $A_t$ does not satisfy strong Markov property. For $1$, suppose that $\tau(\omega)$ is first hitting time of $0$ of $A_t(\omega)$, that is, let $\tau(\omega) = \inf\{t > 0; A_t(\omega) = 0\}$. Then,

$$P_x(A_1 \neq 0, \tau \leq 1) = E_x[P(A_1 \neq 0, \tau \leq 1|\mathcal{F}_t)]$$

$$\leq E_x[P(1 - \tau; A_{\tau}, \{0\}^c)]$$

$$= E_x[P(1 - \tau; 0, \{0\}^c)] = 0,$$  

where $\{0\}^c \triangleq R - \{0\}$. On the other hand, since $P_x(A_1 = 0) = 0$ when $x \neq 0$, we have

$$P_x(A_1 \neq 0, \tau \leq 1) = P_x(\tau \leq 1)$$

$$(2.3)$$

$$= P_x(\omega) : \text{ there exists } t \leq 1 \text{ such that } B_t(\omega) = 0 > 0.$$  

Which contradicts (2.2).  

\hspace{1cm}$\square$

3. Annotation of this example

Remark 3.1. 1. $A_t(\omega)$ is not a homogeneous Markov process since the former two terms of the third equality of (2.1) are both depend on $s$. In fact, if we fix $\omega$, suppose without loss of generality $A_{s_1}(\omega) = x_1, A_{s_2}(\omega) = x_2$ ($s_1 \neq s_2, x_1 \neq x_2$), then

$$P(t; A_{s_1}(\omega), B) = p(t; x_1, B) \neq P(t; s_2, B) = p(t; A_{s_2}(\omega), B);$$

$$\int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} dy = \int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} dy$$

$$\neq \int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} dy = \int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} dy.$$  

So $P(t; A_s, B)$ depends on $s$. But the homogeneity is applied in the second line of (2.2).

2. Since $P(t; A_s(\omega), B)$ depends on $s$, but $P(t; x, B)$ doesn’t, so $P(t; A_s(\omega), B)$ differ from $P(t; x, B)$.

3. The condition $P(A_0(\omega) = 0) \neq 0$ is not used in the proof, that is, it is holds when $P(A_0(\omega) = 0) = 0$ even $\{A_0(\omega) = 0\} = \emptyset$. So we may suppose $P(A_0(\omega) = 0) = 0$. Hence $A_s(\omega)$ and $B_s(\omega)$ are two equivalent even the same processes since $P(A_s(\omega) = B_s(\omega)) = 1$ for every $s \geq 0$, further, $A_s(\omega)$ and $B_s(\omega)$ have the same sample path except the null measurable set $\{A_0(\omega) = 0\}$. Therefore, we think that $A_s(\omega)$ also satisfies strong Markov property since $B_s(\omega)$ is a strong Markov process (See [1, P343] and [7, Theorem 3.10]).
For convenience, the transition probability functions in the example is denoted by

\[
p(s, t; x, B) = \begin{cases} 
\int_B \frac{1}{2\pi(t-s)} e^{-\frac{(x-y)^2}{2(t-s)}} dy & \text{if } x \neq 0, \\
\chi_B(0) & \text{if } x = 0.
\end{cases}
\]

We do not obtain (2.3) contradicts with (2.2) if we use the transition probability functions \( p(s, t; A_s(\omega), B) \) we can prove (2.2) and (2.3) are consistent. (It also verifies indirectly the conclusions about strong Markov property in this paper is valid.) Of which proves as follows:

Let \( P_x(\tau \leq u) = F_x(u); \) \( P_x(\tau \leq u, A_\tau \leq v) = F_x(u, v) \) and \( A_\tau(\{\tau \leq 1\}) = \{(u, v) : \tau(\omega) = u, A_\tau(\omega) = v, \omega \in \{\tau \leq 1\}\}. \) Obviously, \( A_\tau(\{\tau \leq 1\}) = \{(u, v) : 0 \leq u \leq 1, -\infty < v < \infty\}. \) By Tulcea theorem and integral transform theorem, we have

\[
F_x(u, v) = \int_{\{\tau \leq 1\}} P_x(A_\tau \leq v, \tau \leq s) P_x(\tau \leq s) dF_x(s)
\]

\[
= \int_0^u P_x(A_s \leq v) dF_x(s) = \int_0^u \int_{-\infty}^v \frac{1}{\sqrt{2\pi s}} e^{-\frac{(x-y)^2}{2s}} dy dF_x(s).
\]

So we obtain 2th differential formula

\[
d^2F_x(u, v) = \frac{1}{\sqrt{2\pi u}} e^{-\frac{(x-v)^2}{2u}} dv dF_x(u).
\]

Hence,

\[
P_x(A_1 \neq 0, \tau \leq 1) = E_x[E_x(X_\tau_{A_1 \neq 0, \tau \leq 1} | \mathcal{F}(A_u; u \leq \tau)]
\]

\[
= \frac{1}{\sqrt{2\pi u}} e^{-\frac{(x-v)^2}{2u}} dv dF_x(u).
\]

The third equality follows from \( X_{A_1 \neq 0, \tau \leq 1} \) is measurable relative to \( \mathcal{F}(A_u; u \leq \tau) \); the third equality follows from strong Markov property.

When \( x \neq 0 \). By (3.3) we have

\[
P_x(A_1 \neq 0, \tau \leq 1) = \int_{\Omega} \chi_{\{\tau \leq 1\}} \int_{R-\{0\}} \frac{1}{\sqrt{2\pi(t-\tau)}} e^{-\frac{(x-y)^2}{2(t-\tau)}} dy dP(d\omega)
\]

\[
= \int_{\{\tau \leq 1, A_\tau \in R\}} \int_{R-\{0\}} \frac{1}{\sqrt{2\pi(t-\tau)}} e^{-\frac{(x-y)^2}{2(t-\tau)}} dy dP(d\omega)
\]

\[
= \int_{A_\tau(\{\tau \leq 1\})} \int_{R-\{0\}} \frac{1}{\sqrt{2\pi(t-u)}} e^{-\frac{(y-u)^2}{2(t-u)}} dy dF_x(u, v)
\]

\[
= \int_0^1 \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi u}} e^{-\frac{(x-y)^2}{2u}} dy dF_x(u) = P_x(\tau \leq 1), \]

the third equality follows from integrable transform theorem [2, Theorem 3.4.1]; the fourth equality follows from (3.2) and Fubini theorem. So (2.2) and (2.3) are consistent. When \( x = 0 \). Obviously, \( P_x(A_1 \neq 0, \tau \leq 1) = 0 \). On the other hand, note that \( \tau(\omega) = 0 \) for all \( \omega \in \{A_0 = 0\} \), we obtain \( p(\tau; t; A_\tau, \{0\}^c) = X(0)^c = \) \( \chi(0)^c = 0 \). Which is substituted into (3.3) we obtain \( P_x(A_1 \neq 0, \tau \leq 1) = 0 \).
Theorem 3.2. (1) $A_s(\omega)$ and $B_s(\omega)$ are two equivalent processes if $P(A_0(\omega) = 0) = 0$.

(2) $A_s(\omega)$ is a strong Markov process.

Proof. (1) Since $P(A_0(\omega) = 0) = 0$ and $A_s(\omega) = B_s(\omega)$ for every $\omega \notin \{A_0(\omega) = 0\}$, and the definition of two equivalent processes is that if and only if $P(A_s(\omega) = B_s(\omega)) = 1$. From which it follows (1) holds .

(2) Let $\alpha(\omega)$ be an arbitrary stopping time. Since $\mathcal{F}(A_u; u \leq s) \subseteq \mathcal{F}(B_u; u \leq s)$ for every $s \geq 0$, then,

$$\mathcal{F}(A_u; u \leq \alpha) \subseteq \mathcal{F}(B_u; u \leq \alpha).$$

For every $A \in \mathcal{F}(A_u; u \leq \alpha)$, we have

$$\int_A P(A_{t+\alpha} \in B | \mathcal{F}(A_u; u \leq \alpha)) P(d\omega) = \int_A \chi_{\{A_{t+\alpha} \in B\}} \nu(d\omega)$$

where the first equality follows from the definition of expectation; the second equality follows from the definition of $A_s(\omega)$, and $A_{t+\alpha}(\omega) \equiv 0$ for every $\omega \in \{A_0 = 0\}$; the third equality follows from $A\{A_0 \neq 0\} \in \mathcal{F}(A_u; u \leq \alpha) \subseteq \mathcal{F}(B_u; u \leq \alpha)$; the fourth equality follows from strong Markov property of $B_s(\omega)$; the last equality follows from $A_s(\omega) = B_s(\omega)$ for every $\omega \in \{A_0(\omega) \neq 0\}$. Again, $p(\alpha(\omega), t + \alpha(\omega); A_{t+\alpha}(\omega), B)$ is $\mathcal{F}(A_\alpha)$-measurable, and $\mathcal{F}(A_\alpha) \subseteq \mathcal{F}(A_u; u \leq \alpha)$. So by Radon-Nikodym theorem, we have

$$P(\alpha \in B | \mathcal{F}(A_u; u \leq \alpha) = p(\alpha(\omega), t + \alpha(\omega); A_{t+\alpha}(\omega), B), P_{\mathcal{F}(A_u; u \leq \alpha)}-a.e.$$ Which is strong Markov property. \[\square\]

By [1, P343] and [7, Theorem 3.10] we know Brown motion $B_s(\omega)$ is a strong Markov process. But we will prove that there always exists its a version $\bar{p}(s, t; x; B)$ of transition probability functions such that strong Markov property, defined by $\bar{p}(s, t; x; B)$, is not valid.

Theorem 3.3. Let $B_s(\omega)$ be the Brown motion, then there exists a version $\bar{p}(s, t; x; B)$ of transition probability function of $B_s(\omega)$ such that the strong Markov property, defined by $\bar{p}(s, t; x; B)$, is not valid.

Proof. In fact, we may suppose $P(B_0(\omega) = 0) = 0$ since transition probability functions have nothing to do with initial distribution when $s > 0$, we take, for any $0 \leq s < t$,

$$\bar{p}(s, t; B_s, B) = \begin{cases} \chi_{\{B_s \neq 0\}}(\omega) \int_B \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x)^2}{2(t-s)}} \, dy + \chi_{\{B_s = 0\}}(\omega) \chi_B(B_s) & \text{if } s > 0, \\ \int_B \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(y-x)^2}{2(t-s)}} \, dy & \text{if } s = 0. \end{cases}$$
First, we need verify \( \bar{p}(s, t; A_s(\omega), B) \) is \( \mathcal{F}(B_s) \)-measurable. (This step is necessary by the definition of conditional probability, but there is no this step in the proof of given example.) Let

\[
g(x) = \begin{cases} 
X\{x \neq 0\} \int_B \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{2(t-s)}} dy \quad & \text{if } s > 0, \\
\int_B \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{2(t-s)}} dy \quad & \text{if } s = 0.
\end{cases}
\]

Then \( g(B_s(\omega)) = \bar{p}(s, t; B_s(\omega), B) \). Note \( g(x) \) is \( \mathcal{B} \)-measurable, \( B_s(\omega) \) values in \( \mathcal{B} \), so by the measurability of composite function ([2, Theorem 2.2.13]) we know \( \bar{p}(s, t; B_s(\omega), B) \) is \( \mathcal{F}(B_s) \)-measurable. second, we need prove

\[(3.6) \quad P(B_t \in B | \mathcal{F}(B_u; u \leq s)) = \bar{p}(s, t; B_s(\omega), B), \ P_{\mathcal{F}(B_u; u \leq s)} \text{-a.e.} \]

Note \( X\{B_s(\omega)\} X_B(B_s) = 0, \ P_{\mathcal{F}(B_s)} \text{-a.e.} \), then,

\[
\bar{p}(s, t; B_s(\omega), B)) = \int_B \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(B_s(\omega)-y)^2}{2(t-s)}} dy, \ P_{\mathcal{F}(B_s)} \text{-a.e.}
\]

So (3.6) is valid. We obtain \( \bar{p}(s, t; B_s(\omega), B)) \) is a version of transition probability functions of \( B_s(\omega) \) \((\int_B \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(B_s(\omega)-y)^2}{2(t-s)}} dy \) is another version.\). Let \( \tau = \inf(t > 0; B_t = 0) \). If we use strong Markov property, we have

\[
\int_B \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}} dy = p(0, t; x, B) = P_x(B_t \in B) = E_x[E_x[X_{\{B_t \in B\}} | \mathcal{F}(B_u; u \leq \tau)]]
\]

\[
= E_x[X_{\{B_\tau(\omega) \neq 0\}}(\omega) \int_B \frac{1}{\sqrt{2\pi(t-\tau(\omega))}} e^{-\frac{(B_\tau(\omega)-y)^2}{2(t-\tau(\omega))}} dy + X_{\{B_\tau(\omega) = 0\}}(\omega)X_B(B_\tau(\omega))]
\]

\[
= E_x[X_B(0)] = X_B(0),
\]

contradiction. So \( B_s \) is not a strong Markov process. Here the third equality follows from strong Markov property, and \( \tau(\omega) > 0 \) when \( x \neq 0 \); the fourth equality follows from \( \{B_\tau(\omega) \neq 0\} = \emptyset \) and \( \{B_\tau(\omega) = 0\} = \Omega \).

**Remark 3.4.** 1. From above theorems we know that it is not reasonable to define strong Markov property of Markov process by using transition probability functions, because transition probability functions might not be one and only.

2. Although it is not reasonable to define strong Markov property by using transition probability functions, in another paper we will prove that for any Markov process, there always exists a version of transition probability functions of this Markov process such that strong Markov property, defined by this version, holds.

4. **Appendix. Theorems and concepts cited or needed in this paper**

For the convenience to read, we list all theorems cited in this paper.

**Theorem 4.1** (2, Property 2.2.2). Let \( f \) be a mapping from \( \Omega \) to \( E \), \( \mathcal{H} \) be a \( \sigma \)-algebra of \( E \). then \( f^{-1}(\mathcal{H}) \) is a \( \sigma \)-algebra of \( \Omega \).

**Theorem 4.2** (2, Theorem 2.2.13). Let \( \Omega \) be a set, \( (E, \mathcal{E}) \) be a measurable space, \( f \) be a mapping from \( \Omega \) to \( E \). Then \( \varphi \) is a \( f^{-1}(\mathcal{E}) \)-measurable function from \( \Omega \) to \( \hat{R} \) if and only if there exists a \( \mathcal{E} \)-measurable real-valued function \( g \) on \( (E, \mathcal{E}) \) such that \( \varphi = g \circ f \). And if \( \varphi \) is bounded or finite, then \( g \) is bounded or finite.
Theorem 4.3 (2, Theorem 5.2.5). Let \( \xi \) be a random variable defined on the probability space \((\Omega, \mathscr{F}, P)\), \( \mathscr{C} \) be a \( \sigma \)-subalgebra of \( \mathscr{F} \), \( B \) be an arbitrary atom of \( \mathscr{C} \). Then, for any \( \omega \in B \),

\[
E(\xi|\mathscr{C})(\omega) \equiv \text{constant}.
\]

Further, if \( P(B) > 0 \), then

\[
E(\xi|\mathscr{C})(\omega) = \frac{1}{P(B)} \int_B \xi \, dP
\]

for every \( \omega \in B \).

Theorem 4.4 (2, theorem 5.3.1). Let \( \xi \) be a random defined on the probability space \((\Omega, \mathscr{F}, P)\), \( E\xi \) exists, \( f \) be a measurable mapping from \((\Omega, \mathscr{F})\) to \((E, \mathscr{E})\). Then, there exists a \( \mathscr{E} \)-measurable function \( g \), which is \( P_f \)-almost everywhere uniquely determined by \( E(\xi|\mathscr{F}(f)) \), defined on \((E, \mathscr{E})\) such that

\[
E(\xi|\mathscr{F}(f)) = g \circ f, \quad P_f(\cdot, a.e.,
\]

where \( g \) satisfies

\[
\int_A g P_f(dx) = \int_{f^{-1}(A)} \xi P(d\omega)
\]

for every \( A \in \mathscr{E} \), where \( P_f \) is a probability measure derived by \( f \), that is, \( P_f \) satisfies

\[
P_f(A) = P(f^{-1}(A)) \quad \text{for every } A \in \mathscr{E}.
\]

Theorem 4.5 (integrable transform theorem; 2, Theorem 3.4.1). Let \( f \) be a measurable transformation from the measurable space \((\Omega, \mathscr{F})\) to the measurable space \((E, \mathscr{E})\); \( g \) be a measurable function defined on \((E, \mathscr{E})\); \( \mu \) be a measure on \((\Omega, \mathscr{F})\); \( \mu_f \) be a derived measure on \((E, \mathscr{E})\) by \( f \), that is, \( \mu_f(B) \overset{\Delta}{=} \mu(f^{-1}(B)) \) for every \( B \in \mathscr{E} \). Then,

\[
\int_{f^{-1}(B)} g \circ f \, d\mu_f = \int_B g \, d\mu_f,
\]

which means: if one of the two integrals exists, then the other also exists, and the two integrals are equal.

Theorem 4.6 (Radon-Nikodym theorem; 2, Theorem 3.7.6). Let \( \mu \) be a \( \sigma \)-finite measure on \( \sigma \)-algebra \( \mathscr{A} \) of \( \Omega \). If the set function \( \varphi \) defined on \( \mathscr{A} \) is \( \sigma \)-finite and \( \sigma \)-additive and \( \mu \)-continuous, then there exists a \( \mathscr{A} \)-measurable finite function \( f \) defined on \((\Omega, \mathscr{A}, \mu)\) such that \( \varphi \) is the indefinite integral of \( f \) on the measurable space \((\Omega, \mathscr{A}, \mu)\), and \( f \) is \( \mu_{\mathscr{A}} \)-almost surely uniquely determined by \( \varphi \).

Theorem 4.7 (Tulce’a theorem; 2, theorem 5.4.5). Let \((\Omega_n, \mathscr{A}_n), n = 1, 2, \cdots \) be sequence of measurable spaces. Set \( \Omega^{(n)} = \prod_{k=1}^n \Omega_k \), \( \mathscr{A}^{(n)} = \prod_{k=1}^n \mathscr{A}_k \), \( \mathscr{A}^{(\infty)} = \bigcap_{k=1}^\infty \mathscr{A}_k \). Let \( P_n(\omega_1, \cdots, \omega_{n-1}, A_n), (\omega_1, \cdots, \omega_{n-1}, A_n) \in \Omega^{(n-1)} \times \mathscr{A}_n, n = 2, 3, \cdots \) be the transition probabilities; \( P_1(A), A \in \mathscr{A}_1 \) be the probability on \( \mathscr{A}_1 \). Then there exists only one probability measure \( P^{(\infty)} \) on \( \mathscr{A}^{(\infty)} \) such that

\[
P^{(\infty)}(C(B^{(n)})) = P^{(n)}(B^{(n)})
\]

and

\[
P^{(n)}(B^{(n)}) = \int_{\Omega_1} \cdots \int_{\Omega_n} X_{B^{(n)}}(\omega_1, \cdots, \omega_n) P_n(\omega_1, \cdots, \omega_{n-1}, d\omega_n) \cdots P_1(d\omega_1).
\]

Here \( C(B^{(n)}) \) indicates the cylinder set based on \( B^{(n)} \), \( B^{(n)} \in \mathscr{A}^{(n)} \).
Theorem 4.8 (Fubini’s theorem; 2, Theorem 4.2.1). Let \((\Omega_i, \mathcal{A}_i, \mu_i), i = 1, 2\) be two \(\sigma\)-finite measurable spaces, \(f\) be nonnegative \(\mathcal{A}_1 \times \mathcal{A}_2\)-measurable function. Then
\[
\int_{\Omega_1 \times \Omega_2} f \, d\mu_1 \times \mu_2 = \int_{\Omega_1} \left( \int_{\Omega_2} f(\omega_1, \omega_2) \, d\mu_2(\omega_2) \right) \, d\mu_1(\omega_1)
\]
\[
= \int_{\Omega_2} \left( \int_{\Omega_1} f(\omega_1, \omega_2) \, d\mu_1(\omega_1) \right) \, d\mu_2(\omega_2).
\]

References

Department of Economic, Hainan University, P.R. China, 570228
E-mail address: tanyou01@163.com