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Non-Gaussianity in Axion $N$-flation Models

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We study perturbations in the multifield axion $N$-flation model, taking account of the full cosine potential. We find significant differences from previous analyses which made a quadratic approximation to the potential. The tensor-to-scalar ratio and the scalar spectral index move to lower values, which nevertheless provide an acceptable fit to observation. Most significantly, we find that the bispectrum non-Gaussianity parameter $f_{\text{NL}}$ may be large, typically of order 10 for moderate values of the axion decay constant, increasing to of order 100 for decay constants slightly smaller than the Planck scale. Such a non-Gaussian fraction is detectable. We argue that this property is generic in multifield models of hilltop inflation.

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Much focus has been placed lately on the discovery potential of cosmic non-Gaussianity in the statistics of primordial perturbations. The Wilkinson Microwave Anisotropy Probe has already set interesting limits [1]. The Planck satellite, now taking data, will improve these significantly, reaching a sensitivity to the non-Gaussian parameter $f_{\text{NL}}$ of around 5. Discovery of non-Gaussianity would open a new arena of cosmological observations particularly suited to probing early Universe physics.

Present ideas in fundamental physics suggest there may be many scalar fields which can influence the early Universe, including inflation. $N$-flation [2] uses many string axions to provide a realization of the “assisted-inflation” phenomenon [3], in which a collection of scalar fields cooperatively support inflation even if their potentials are individually too steep. The phenomenology of such models (see also Ref. [4]) links fundamental physics and upcoming cosmological observations.

Previous $N$-flation studies have assumed that all relevant fields are close to their minima and can be described by quadratic potentials. For axions the full potential is trigonometric and we find that the quadratic approximation is unreliable. Even for identical potentials, the condition for stable coevolution of the fields is violated near the hilltop [5]. Therefore, fields in this region evolve on divergent trajectories. Accounting for this divergence by retaining the full potential leads to two very significant changes. The predicted scalar spectral index and tensor-to-scalar ratio, $r$, are reduced. This remains compatible with existing observations but may leave $f_{\text{NL}}$ undetectable. More importantly, $f_{\text{NL}}$ is predicted to be large, and very plausibly within the range of future probes.

This unexpectedly large non-Gaussianity is a genuine multifield phenomenon. It is a consequence of the diverging trajectories near the hilltop, implied by a negative $\eta$ parameter of order unity or larger. In single-field models, potentials of this form lead to a density perturbation with a spectral index $n$ in conflict with observation. The assisted-inflation mechanism reduces $1 - n$ to an acceptable value, but leaves $f_{\text{NL}}$ dominated by the contribution of the field closest to the peak.

The model.—The axion $N$-flation model is based on a set of $N_f$ uncoupled fields, labeled $\phi_i$, each with a potential [2]

$$V_i = \Lambda_i^4 (1 - \cos \alpha_i),$$

where $\alpha_i = 2\pi \phi_i / f_i$ and $f_i$ is the $i$th axion decay constant. In a more general model, couplings may exist between the fields, but we will not consider these. The mass of each field in vacuum satisfies $m_i = 2\pi \Lambda_i^4 / f_i$, and the angular field variables $\alpha_i$ lie in the range $(-\pi, +\pi)$. Without loss of generality, we will set initial conditions with all $\alpha_i$ positive. If only a single field is present, this model is known as natural inflation [6].

Calculation of the observables $n$, $r$, and $f_{\text{NL}}$ makes use of the $\delta N$ formula [7], which considers how the total number of e-folds of expansion $N$ is modified by field perturbations. We define slow-roll parameters for each field as

$$\epsilon_i = \frac{M_P^2}{2} \left( \frac{V_i'}{V_i} \right)^2,$$

where $M_P \equiv (8\pi G)^{-1/2}$ is the reduced Planck mass, a prime denotes the derivative of a function with respect to its argument, and no summation over $i$ is implied. The global slow-roll parameter $\epsilon = -\dot{H}/H^2$ can be written as a weighted sum $\epsilon = \sum_i (V_i/V)^2 \epsilon_i$, in which each field contributes according to its share of the total energy density. We must have $\epsilon < 1$ during inflation.

We work in the horizon-crossing approximation, in which the dominant contribution to each observable is assumed to arise from fluctuations present only a few e-folds after horizon exit of the wave number under discussion. After smoothing the universe on a superhorizon...
scale somewhat smaller than any scale of interest, the horizon-crossing approximation becomes valid whenever the ensemble of trajectories followed by smoothed patches of the universe approaches an attractor. We suppose that inflation exits gracefully, with each field settling into the minimum of its potential. The horizon-crossing formulas will then be a reasonable approximation. Using this method, and conventional definitions for each observable parameter [8], we find

$$\mathcal{P}_\xi = \frac{H^2}{4\pi^2} \sum_i N_i \epsilon_i = \frac{H^2}{8\pi^2 M_P^2} \sum_i \frac{1}{\epsilon_i^2},$$

(3)

$$n - 1 = -2e_* - \frac{8\pi^2}{3H^2} \sum_j \frac{\Lambda_j}{f_j} \epsilon_j / \sum_i \epsilon_i^2,$$

(4)

$$r = \frac{2}{\pi^2 \mathcal{P}_\xi} \frac{H^2}{M_P^2} = \frac{16}{2 \sum_i \frac{1}{\epsilon_i^2}}.$$  

(5)

$$6f_{NL} = \sum_{ij} N_i N_j N_{ij} \left( \sum_i N_i k_i^2 \right)^2 = \frac{r^2}{128} \sum_i \frac{1}{\epsilon_i^2} 1 + \cos\alpha_i,$$

(6)

where $N_i$ and $N_{ij}$ are, respectively, the first and second derivatives of $N$ with respect to the fields, and $*$ indicates evaluation at horizon crossing [determined by Eq. (7) below]. In writing Eq. (6) any intrinsic non-Gaussianity among the field perturbations at horizon crossing has been neglected, a good approximation provided $f_{NL} > 1$ [9,10]. Our sign convention for $f_{NL}$ matches the Wilkinson Microwave Anisotropy Probe team [1], and the non-Gaussianity is predicted to be of local type. The observed amplitude of perturbations is obtained by adjusting the $\Lambda_i$ to give an appropriate value of $H_*$. Under a quadratic approximation to each potential, it can be shown that Eqs. (5) and (6) recover their single-field values of order $1/N$. Making $f_{NL}$ undetectably small. The spectral index can be shown to be less than its single-field value $1 - 2/N$, with equality only in the equal-mass case. Its value for a given choice of parameters must be computed numerically [13]. However, we will see that these results all change whenever our initial conditions populate the hilltop region.

$N$-flation perturbations.—Equations (3)–(6) apply for any choice of $\Lambda_i$ and $f_i$. We restrict attention to the case where all fields have the same potential, which already captures the interesting phenomenology. A broader investigation will be published elsewhere. The scale $\Lambda \equiv \Lambda_i$ is fixed from the observed amplitude of $\mathcal{P}_\xi$, leaving $f \equiv f_i$ and $N_f$ adjustable parameters. The initial conditions are drawn randomly from a uniform distribution of angles $\alpha_i$, with several realizations to explore the probabilistic spread. From these two parameters we predict the observables $n$, $r$, and $f_{NL}$.

There are two constraints. First, we require sufficient e-foldings. For a given set of initial angles $\alpha_i$, and ignoring a small correction from the location of the end of inflation, one finds

$$N_{\text{tot}} \approx \sum_i \frac{f_i}{2 M_P^2} \ln \frac{2}{1 + \cos\alpha_i} \simeq \ln^2 \frac{f_i^2}{2 M_P^2} N_f,$$

(7)

where in the second equality we have replaced $N_{\text{tot}}$ with its expectation value by averaging over $\alpha_i$. Equation (7) is replicated to high accuracy in numerical simulations. For a given $f$ it determines the minimum number of fields required for sufficient inflation, typically several hundred or more. There is no similar constraint from the spectral index. When $N_{\text{tot}} = N_{\alpha}$, the $\alpha_i$ are uniformly distributed and $(n - 1) = -5 \ln 2/N_{\alpha}$, independent of $f$ and $N_f$. This tilt is observationally acceptable. For larger $N_f$ the spectral index approximately satisfies Eq. (8) below.

Second, a key motivation of the $N$-flation model was to obviate the requirement for super-Planckian field values, which are invoked in many single-field models. If one literally imposes $|\phi| < M_P$, this requires $f_i < 2 M_P$ for each $i$. However, it would be reasonable to regard this condition as approximate and not mandatory.

The $\epsilon_i$ approach zero for fields close to the hilltop, so each summation in Eqs. (3)–(6) is dominated by those fields with the smallest $\epsilon_i$. Suppose some number $\tilde{N}$ of fields have roughly comparable $\epsilon_i$, of order $\tilde{e}$. The observable parameters have different scalings with $\tilde{N}$. The spectrum $\mathcal{P}_\xi$ scales like $\tilde{N}$ copies of a single-field model with slow-roll parameter $\tilde{e}$, whereas $r$ is reduced by a factor $\tilde{N}$ compared to its value in the same single-field model. The spectral index can be written exactly (within slow roll) in terms of a single sum coming from $H_*$,

$$n - 1 = -2e* - \frac{8\pi^2}{3H^2} \left( \frac{M_P}{f} \right)^2 / \sum (1 - \cos\alpha_i),$$

(8)

and is independent of $\tilde{N}$. It becomes close to $-2e_*$ when the denominator is of order $10^3$. This is the standard assisted-inflation mechanism. Most importantly, $f_{NL}$ has the approximate behavior

$$6f_{NL} = \frac{2\pi^2}{\tilde{N}} \left( \frac{M_P}{f} \right)^2,$$

(9)

which is independent of $\tilde{e}$ if the dominant fields are sufficiently close to the hilltop. $N$-flation has lifted the single-field consistency condition $f_{NL} = -(5/12)(n - 1)$ [9,10], which prevents single-field models generating large non-Gaussianity without violating observational bounds on $n$.

Where the summations in Eqs. (3)–(6) are dominated by a single field, this formula shows that $f_{NL}$ can become rather large, scaling as $(M_P/f)^2$. For $f = M_P$, we find $f_{NL} \approx 16.4$; a non-Gaussian fraction of this magnitude should be visible to the Planck satellite. It is even possible to achieve $f_{NL} \approx 100$ for $f \approx 0.4 M_P$, though then $N_f$ must
be very large to gain sufficient e-foldings. If \( f_{\text{NL}} \approx 50 \) it may be more profitable for Planck to search for non-linearity in the trispectrum [14], for which estimates in the quadratic approximation were given in Ref. [15]. We defer a full analysis of the trispectrum to future work but note that the trispectrum equivalents of Eq. (9) are, in conventional notation [16], \( \tau_{\text{NL}} = (4\pi^4/\mathcal{N}^2)(M_p^4/f^4) \) and \( (54/25)\alpha_{\text{NL}} = (8\pi^4/\mathcal{N}^2)(M_p^4/f^4) \).

The expectations described above are borne out in numerical calculations. In Fig. 1 we show model predictions in the \( n-r \) plane, averaged over several realizations of the initial conditions. We see \( n \) and \( r \) are only weakly dependent on the model parameters (though there is significant dispersion amongst realizations, not shown here), with the choice of \( N_s \) being the principal determinant of \( n \).

In Fig. 2 we plot \( f_{\text{NL}} \) as a function of \( N_f \) for \( f = M_p \), with ten realizations at each \( N_f \). This clearly shows the expected maximum, which is nearly saturated in cases where a single field dominates the summations. In cases where several fields contribute significantly to the sums in Eqs. (3)–(6), the non-Gaussian fraction is reduced.

Equations (8) and (9) clarify the origin of large \( f_{\text{NL}} \) in this model. The cooperative effect of the \( N \)-flation mechanism does not enhance the non-Gaussian signal. Indeed, \( f_{\text{NL}} \) is suppressed by the central limit theorem, where \( N \gg 1 \) fluctuations contribute equally to the curvature perturbation. Nor does the large effect arise from a singularity in the e-folding history, \( N_s \), as a function of its initial angles \( \alpha_i \). Although Eq. (7) is singular in the limit \( \alpha_i \to \pi \), its Taylor expansion is trustworthy unless \( |\alpha_i - \pi| < r^{1/2}(f_i/M_p) \). The observed magnitude of \( \mathcal{P}_i \) requires \( |\alpha_i - \pi| \approx r^{1/2}(f_i/M_p) \) for each field, so a breakdown of the Taylor expansion cannot become relevant unless at least one \( f_i \) is a few orders of magnitude less than the Planck scale, of order \( (f_i/M_p)^4 \ll \mathcal{P}_i \). These constraints additionally imply that we do not trespass on any region of field space where quantum diffusion competes with classical motion.

Instead, the large \( f_{\text{NL}} \) derives from a generic dispersive effect present in any hilltop potential. Measuring the displacement of \( \phi_i \) from the hilltop by \( \delta_i \), each potential can be approximated in its vicinity by \( V_i = 2\Lambda_i^2(1 + \eta_i\delta_i^2/2M_p^2) \), where \( \eta_i < 0 \) satisfies

\[
\eta_i = M_p^2 V_i'/V_i = -2\pi^2 \left( M_p^2 f_i^2 \right)^2.
\]

These potentials are tachyonic. Fields close to the hilltop remain almost stationary, while fields further away are ejected downhill. This process typically leaves a few fields on top of the hill, which have small \( \epsilon_i \) and dominate the sums in Eqs. (3)–(6). It seems clear that this behavior is generic for any \( N \)-flation model constructed using hilltop potentials. The few fields remaining in the vicinity of the hilltop each generate contributions to the curvature perturbation with third moment \( (6/5)f_{\text{NL}} \approx -\eta_i \cdot 19 \).

Accounting for suppression arising from the central limit theorem, we recover the approximate expression (9). For a general hilltop potential, well-rehearsed arguments lead us to expect \( |\eta| \approx 1 \) and therefore \( f_{\text{NL}} \approx 1 \). In a single-field model this is the "\( \eta \) problem." In an \( N \)-flation model, it is a generic expectation of enhanced non-Gaussianity. Even larger yields are possible in some models, including our case, if it is possible to achieve \( |\eta| \gg 1 \) while preserving technical naturalness.

Conclusions.—We have described a new mechanism for generating observably large cosmic non-Gaussianity, based

![FIG. 1 (color online). Predictions in the \( n-r \) plane, averaged over realizations, for various values of \( f \) between 0.4M_p and 2M_p and of \( N_f \) between 464 and 10000, all giving sufficient inflation. The black (left) cluster of points takes \( N_s = 50 \) and the red (right) cluster \( N_s = 60 \). The quadratic expansion predicts \( r = 8/N_s \), far off the top of this plot. The region right of the line is within the WMAP7 + BAO + \( H_0 \) 95% confidence contour [1].](image1)

![FIG. 2 (color online). Predicted non-Gaussianity, \( \frac{1}{2} f_{\text{NL}} \), for \( f = M_p \) and \( N_s = 50 \). The error bars are on the mean over realizations (not the standard deviation). Here the maximum achievable value of \( \frac{1}{2} f_{\text{NL}} \) is \( 2\pi^2 \approx 20 \), almost saturated in some realizations. The significant spread is due to initial condition randomness with typical mean values being around half the maximum achievable value, and no discernible trend with \( N_f \).](image2)
on the strongly dispersive dynamics of fields in a hilltop region. In a multifield context such as the axion \( N \)-flation model, assisted inflation can yield a viable spectral index without a major dilution of the non-Gaussianity. As compared to the quadratic potential approximation to \( N \)-flation, we found a substantial decrease in \( r \), a modest increase in \( 1 - n \), and a substantial increase in \( f_{\text{NL}} \). These changes will happen whenever initial conditions have a significant probability of populating the hilltop region, such as the uniform (in field angle) initial conditions we chose.

 Searches have previously been made for models which achieve \(|f_{\text{NL}}| \gg 1\) while preserving slow roll during inflation [17]. The \( N \)-flation model is of this type, but offers several advantages. The non-Gaussian fraction is naturally bounded above, so that \( f_{\text{NL}} \) cannot become arbitrarily large. Therefore, our predictions do not depend on a sudden exit from inflation, e.g., triggered by a hybrid transition, to prevent \( f_{\text{NL}} \) from growing to an unacceptable value. Equally important, our large signal does not derive from a singularity of the e-folding history \( N \), as a function of its initial conditions. This means we can rely on a perturbative expansion. We can simultaneously satisfy observational constraints on the spectral index and tensor fraction. Moreover, this result seems generic. Inflation is self-replicating on top of the hill, sometimes described as "topological inflation" [18]. Coupled with the dispersion of trajectories originating from the vicinity of the hilltop, this implies that large non-Gaussianity may not be uncommon over a landscape of scalar field vacua.

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